1.1

Relations

Learning Objectives:

- To define a relation from a set to another set and study its representations.
- To study different types of relations
- To study the properties of relations on a given set
  AND
- To practice the related problems

Representation of a Relation

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any two sets, not necessarily different.
A (binary) relation from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a subset of $A \times B$.
If R is a relation from A to B, then R is a set of ordered pairs from $A \times B$, where each first and second element of the ordered pairs of R come from A and B respectively. That is, for each pair $a \in A \text{ and } b \in B$, exactly one of the following is true:
$(a, b) \in R$; we then say ***a is R-related to b***, written $\boldsymbol{a\,R\,b}$
$(a, b) \notin R$; we then say ***a is not R-related to b***

Sometimes R is a relation from a set A to itself, that is, R is a subset of $A^2 = A \times A$. In such a case, we say that ***R is a relation on A***.

A relation can be represented in *roster form*: We represent the relation by a set of ordered pairs, which satisfy a given relation.

Example: Let $A = \{1,4\}, B = \{1,2\}$ and let R mean *is the square of*. Express this relation in roster form.
Solution:
Here, $A \times B = \{(1,1), (1,2), (4,1), (4,2)\}$
Then, $R = \{(1,1), (4,2)\}$

The relation has an alternative representation known as *set builder form*. In set builder form, the relation from $A \text{ to } B$ is represented as:

$$\{(x, y) | x \epsilon A,\ y \epsilon B, x - - - - y\},$$

where the dashed line is the rule which associates $x \text{ and } y$.

Example: Let $A = \{1,2,3\}, B = \{4,5\}$ and let R mean *is less than*. Express this relation in the set-builder form.
Solution:
Here, $A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$
Then, $R = \{(x, y) \mid x \in A, y \in B, x < y\}$

Domain and Range

The ***domain*** of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R, and so it is a subset of A. The ***range*** of R is the set of all second elements, and so it is a subset of B. It is a convention to call the set B as ***co-domain*** of R.

Example: Let $A = \{1,3,4,5,7\}, B = \{2,4,6,8\}$ and R be the relation *is one less than* from $A \text{ to } B$. Find the domain, and range of R.
Solution: From the set $A \times B$, we find that

$$R = \{(1,2), (3,4), (5,6), (7,8)\}$$

Hence, the domain of R = {1,3,5,7} and
range of R = {2,4,6,8}

Theorem: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are finite sets with $\boldsymbol{m}$ and $\boldsymbol{n}$ elements respectively, then the number of relations from $\boldsymbol{A}$ to $\boldsymbol{B}$ is $\boldsymbol{2^{mn}}$.

Proof:

$$n(A) = m, n(B) = n \Rightarrow n(A \times B) = n(A) \cdot n(B) = mn$$

We have, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.
Therefore, the number of relations from A to B
$=$ the number of subsets of $A \times B = 2^{n(A \times B)} = 2^{mn}$

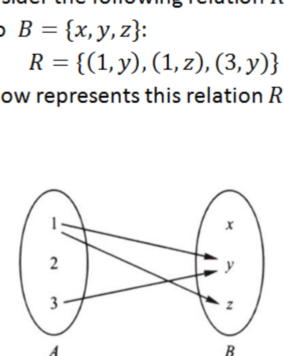
Geometrical representation

There are a number of ways of representing the relations. Let S be a relation on the set $\boldsymbol{R}$ of real numbers; that is, let S be a subset of $\boldsymbol{R}^2 = \boldsymbol{R} \times \boldsymbol{R}$. Since $\boldsymbol{R}^2$ can be represented by the set of points in the plane, we can represent S by plotting those points in the plane which belong to S. This pictorial representation is called the ***graph of S***. Frequently, the relation S consists of all ordered pairs of real numbers which satisfy an equation, and in this case we identify the relation with the equation.

Example: Consider the relation S defined by the equation

$$x^2 + y^2 = 25$$

The relation S consists of all ordered pairs (x_0, y_0) which satisfy the given equation. The graph of the equation is a circle having its center at the origin and radius 5.



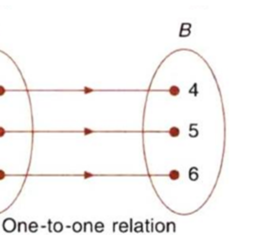
$x^2 + y^2 = 25$

Suppose $A \text{ and } B$ are finite sets. A relation R from $A \text{ to } B$ may be represented by an arrow diagram.

Example: Consider the following relation R from $A = \{1,2,3\}$ to $B = \{x, y, z\}$:

$$R = \{(1, y), (1, z), (3, y)\}$$

The figure below represents this relation R by an arrow diagram.



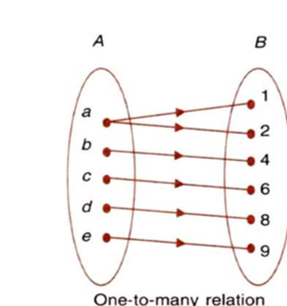
Types of Relations

A relation from A to B is said to be ***one to one*** if one member of A is associated with one member of B only.

The relation

$$R = \{(p, 4), (q, 5), (r, 6)\}$$

is one-to-one, as shown below.



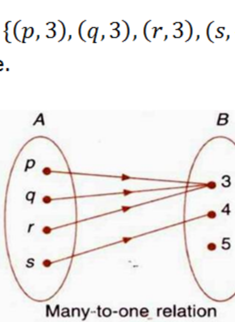
One-to-one relation

A relation from A to B is said to be ***one to many*** if one member of A is associated with more than one member of B.

The relation

$$R = \{(a, 1), (a, 2), (b, 4), (c, 6), (d, 8), (e, 9)\}$$

is one-to-many.



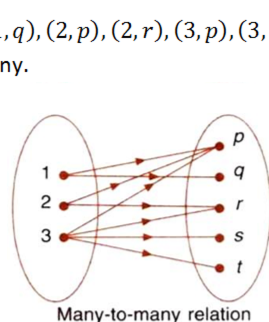
One-to-many relation

A relation from A to B is said to be ***many to one*** if many elements of A are associated with the same element of B.

The relation

$$R = \{(p, 3), (q, 3), (r, 3), (s, 4)\}$$

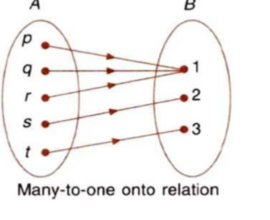
is many-to-one.



Many-to-one relation

A relation from A to B is said to be ***many to many*** if many elements of A are associated with the same element in B, and also each member of A is associated with more than one member of B.

The relation
$R = \{(1, p), (1, q), (2, p), (2, r), (3, p), (3, r), (3, s), (3, t)\}$
is many-to-many.



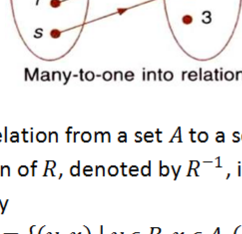
Many-to-many relation

A relation from A to B is called ***onto*** if every element of B has a corresponding element in A.

The relation

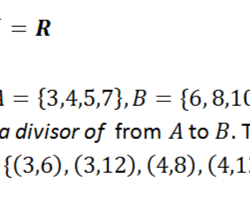
$$R = \{(p, 1), (q, 1), (r, 1), (s, 2), (t, 3)\}$$

is many-to-one and onto.



Many-to-one onto relation

A relation from A to B is called **into** if every element of B does not have a corresponding element in A.



Many-to-one into relation

Let R be the relation from a set A to a set B, then the ***inverse*** relation of R, denoted by R^{-1}, is a relation from B to A defined by

$$R^{-1} = \{(y, x) \mid y \epsilon B, x \epsilon A, (x, y) \epsilon R\}$$

It is noted that the domain of R^{-1} is the range of R and range of R^{-1} is the domain of R.
Note: $\left(\boldsymbol{R}^{-1}\right)^{-1} = \boldsymbol{R}$

Example: Let $A = \{3,4,5,7\}, B = \{6, 8,10,11,12\}$, and R be the relation *is a divisor of* from A to B. Then

$$R = \{(3,6), (3,12), (4,8), (4,12), (5,10)\}.$$

Find R^{-1}.
Solution: Domain of $R = \{3,4,5\}$ and range of R = {6,8,10,12}. Therefore, R^{-1} is a relation from B to A given by

$$R^{-1} = \{(6,3), (12,3), (8,4), (12,4), (10,5)\}$$

Domain of R^{-1} = {6,8,10,12}, Range of R^{-1} = {3,4,5}.
R^{-1} is the relation *is divisible by* from B to A.

Properties of Relations

A relation R on a set A is called ***reflexive*** if each member of A is in relation to itself, i.e., xRx, for all $x \epsilon A$. Thus, it contains ordered pairs (x, x) for all $x \in A$.

Example: If L be the set of all lines in a plane and R means *is parallel to*, then any line $x \epsilon L$ is parallel to itself. Therefore, xRx is true for every line $x \epsilon L$. Thus R is reflexive.

A relation R on a set A is called ***symmetric*** if xRy then yRx i.e., $(x, y) \epsilon R \Rightarrow (y, x) \epsilon R$

Example: If L is the set of all lines in a plane and R stands for *is perpendicular to*, then if a line x is perpendicular to the line y, then y is also perpendicular to x, i.e., $xRy \Rightarrow yRx$. Thus R is a symmetric relation.

A relation R on a set A is called ***transitive*** if xRy and $yRz \Rightarrow xRz$, i.e.,$(x, y) \epsilon R, (y, z) \epsilon R \Rightarrow (x, z) \epsilon R$
In other words, if x is related to y, and y is related to z, then x is related to z.

Example: Let L be a set of all lines in a plane and R stands for *is parallel to*. If a line a is parallel to b and if b is parallel to c, then a is parallel to c. Thus,

$$aRb \text{ and } c \Rightarrow aRc.$$

Hence R is transitive.

A relation R on a set A is said to be an ***equivalence relation*** if R is reflexive, symmetric and transitive. That is, a relation R on a set A is said to be an equivalence relation on A if it satisfies the following conditions:

i. xRx for all $x \epsilon A$
ii. $xRy \Rightarrow yRx; x, y \epsilon A$
iii. xRy and $yRz \Rightarrow xRz\,; x, y, z \epsilon A$.

The symbol $\sim$ is used for an equivalence relation.

Example: In the set T of all triangles in a plane, show that the relation of congruence is an equivalence relation.
Solution: Here, R stand for *is congruent to*. Since a triangle $x \epsilon T$ is congruent to itself, xRx is true for all x. Hence, the relation is reflexive.
If a triangle $x \epsilon T$ is congruent to the triangle $y \epsilon T$, then y is congruent to x, i.e., $xRy \Rightarrow yRx$. Hence, the relation is symmetric.
If $x, y, z \epsilon T$ and if x is congruent to y and y is congruent to z, then x is congruent to z. Thus, xRy and $\Rightarrow xRz$. Hence, the relation is transitive.
Thus, the relation of congruence in T is an equivalence relation on T.

Example:
The relation *is parallel to* is an equivalence relation in the set L of all lines in a plane.
Example:
The relation is similar to is an equivalence relation in the set T of all triangles in a plane.

Definition:
Let m be a positive integer. If a and b are integers, then ***a is congruent to b modulo m*** return as $a \equiv b (mod\ m)$, if m divides $a - b\ i.e., a - b$ is a multiple of m.
The relation *is congruent modulo* m is an equivalence relation on Z, the set of integers.

1.2

Functions

Learning Objectives:

- To define a function from a set into another set and to view functions as relations
- To study the composite functions
  AND
- To solve the related problems

The terms *map, mapping, transformation* are also used as alternative names for the function. The choice of which word is used in a given situation is usually determined by tradition.

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a ***function from A to B***. The set A is called the ***domain*** of the function, and the set B is called the ***co-domain***. Let f denote a function from A to B. Then we write

$$\boldsymbol{f: A \rightarrow B}$$

which is read: ***f is a function from A into B***, or ***f maps A into B***.

Suppose $f: A \rightarrow B$ and $a \epsilon A$. Then $f(a)$, read "f of a", will denote the unique element of B which f assigns to a. This element $\boldsymbol{f(a)}$ in B is called the ***image of a under f*** or the ***value of f at a***. We also say that f *sends* or *maps* a into $f(a)$. The set of all such image values is called the ***range*** or ***image*** of f, and it is denoted by $\boldsymbol{Ran(f)}$, $\boldsymbol{Im(f)}$ or $\boldsymbol{f(A)}$. That is,

$$\boldsymbol{Im(f) = \{f(a) \mid a \epsilon A\}}$$

Clearly, $Im(f) \subseteq B$.

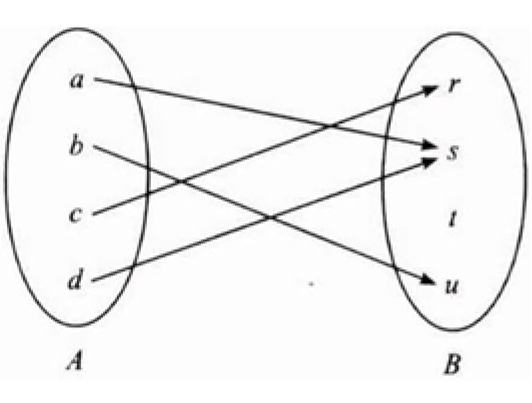
Frequently, a function can be expressed by means of a mathematical formula. For example, consider the function which sends each real number into its square. We may describe this function by $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ as $f(x) = x^2$. Here, x is called a ***variable*** of the function f.

Suppose $f: A \rightarrow B$. If A' is a subset of A, then $f(A')$ denotes the set of images of elements in A'; and if B' is a subset of B, then $f^{-1}(B')$ denotes the set of elements of A each, whose image belongs to B'. That is,

$$\boldsymbol{f(A') = \{f(a) \mid a \epsilon A'\}} \text{ and}$$
$$\boldsymbol{f^{-1}(B') = \{a \epsilon A \mid f(a) \epsilon B'\}}$$

We call $f(A')$ the ***image of A'***, and we call $f^{-1}(B')$ the ***inverse image*** or ***pre- image of B'***.

Example:
The figure below defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$



From the figure,

$f(a) = s, \quad f(b) = u, \quad f(c) = r, \quad f(d) = s$

The image of f is the set $\{r, s, u\}$. We note that t does not belong to the image of f because t is not the image of any element of A under f.

Identity Function

Consider any set A. Then there is a function from A into A which sends each element of A into itself. It is called the ***identity function on A*** and it is usually denoted by $\boldsymbol{I_A}$. In other words, the identity function $\boldsymbol{I_A: A \rightarrow A}$ is defined by $\boldsymbol{I_A(a) = a}$ for every element $a \epsilon A$.

Functions as Relations

There is another point of view from which functions may be considered. First of all, every function $f: A \rightarrow B$ gives rise to a relation from A to B called the *graph* of f and defined by

$$\text{Graph of } f = \{(a, b) \mid a \epsilon A, b = f(a)\}$$

Two functions $f: A \rightarrow B$ and $g: A \rightarrow B$ are defined to be ***equal***, written $f = g$, if $f(a) = g(a)$ for every $a \epsilon A$; that is, if they have the same graph. Accordingly, we do not distinguish between a function and its graph. We study the graphs of several functions in a future module.

Now, such a graph relation has the property that each $a\epsilon A$ belongs to a unique ordered pair (a, b) in the relation. Consequently, one may equivalently define a function as follows:

Definition: A function $f: A \rightarrow B$ is a relation from A *to* B (i.e., a subset of $A \times B$) such that each $a \epsilon A$ belongs to a unique ordered pair (a, b) in f.
The defining condition of a function, that each $a \epsilon A$ belongs to a unique pair (a, b) in f, is equivalent to the geometrical condition of each vertical line intersecting the graph in exactly one point.

Example: Consider the relation f from $A = \{a, b, c, d\}$ to $B = \{s, u, r\}: f = \{(a, s), (b, u), (c, r), (d, s)\}$. Ascertain whether this is a function.

Solution:
The given relation is a function $f: A \rightarrow B$ with the domain $A = \{a, b, c, d\}$ and the range = $\{s, u, r\}$, since each member of A appears as the first coordinate in exactly one ordered pair in f.

Example: Consider the following relations on the set $A = \{1,2,3\}$.

$$f = \{(1,3), (2,3), (3,1)\},$$
$$g = \{(1,2), (3,1)\},$$
$$h = \{(1,3), (2,1), (1,2), (3,1)\}$$

Ascertain whether each relation is a function.

Solution:

The relation f is a function from A into A, since each member of A appears as the first coordinate in exactly one ordered pair in f; here $f(1) = 3, f(2) = 3, f(3) = 1$.

The relation g is not a function from A into A since $2 \epsilon A$ is not the first coordinate of any pair in g and so g does not assign any image to 2.

The relation h is not a function from A into A since $1 \epsilon A$ appears as the first coordinate of two distinct ordered pairs in h, (1,3) and (1,2). If h is to be a function it cannot assign both 3 and 2 to the element $1 \epsilon A$.

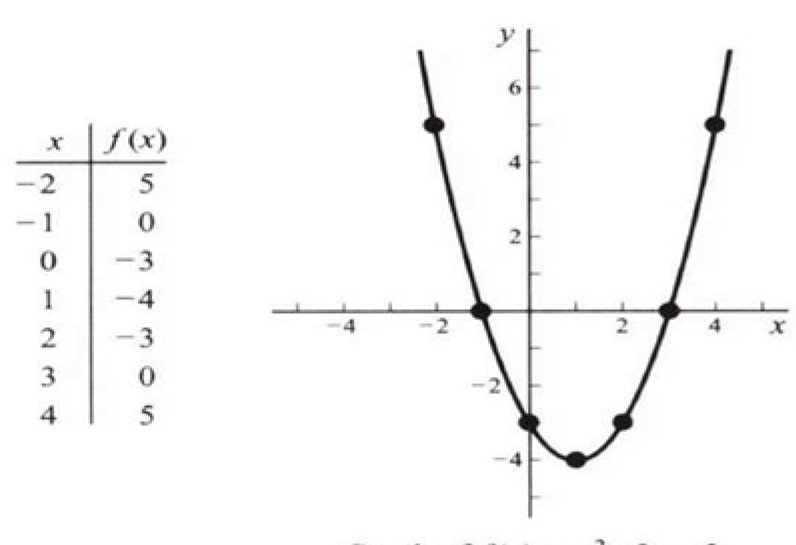
Polynomial function

Consider the function $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers. This is called a ***polynomial function***. Since $\boldsymbol{R}$ is an infinite set, the graph of such a function can be drawn by first plotting some of its points and then drawing a smooth curve through these points. For example, the graph of $f(x) = x^2 - 2x - 3$ is obtained by plotting the following points and then drawing a smooth curve through them.

x	-2	-1	0	1	2	3	4
$f(x)$	5	0	-3	-4	-3	0	5



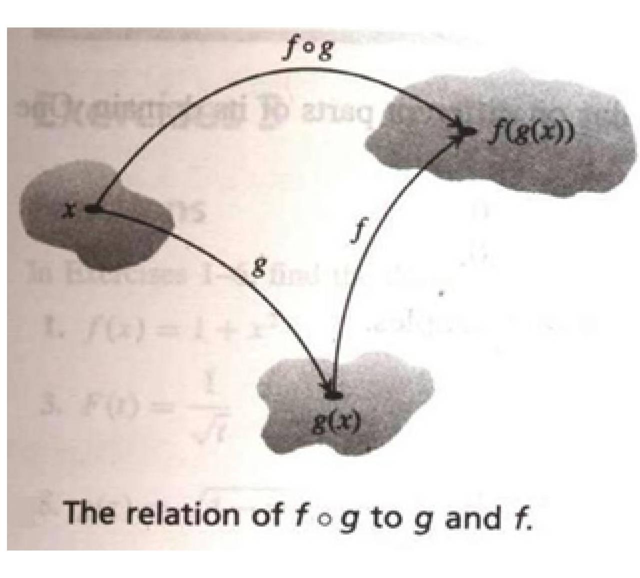
Graph of $f(x) = x^2 - 2x - 3$

Composite Functions

Composition is a method for combining functions. If f and g are functions, the composite function $f \circ g$ ("f *circle* g") is defined by

$$\boldsymbol{(f \circ g)(x) = f(g(x))}$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.
The definition says that two functions can be composed when *the range of the first lies in the domain of the second.*



The relation of $f \circ g$ to g and f.

To evaluate the composite function$(f \circ g)(x)$, we first find $g(x)$ and second find $f(g(x))$. To evaluate the composite function $(g \circ f)$, we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $(g \circ f)$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.
The functions $f \circ g$ and $g \circ f$ are usually quite different.

Example
If $f(x) = \sqrt{x}$ *and* $g(x) = x + 1$, find

a) $(f \circ g)(x)$
b) $(g \circ f)(x)$
c) $(f \circ f)(x)$
d) $(g \circ g)(x)$

Solution:

Composite	***Domain***
a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$	$[0, \infty)$
d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$(-\infty, \infty)$

1.3. Types of Functions

Learning Objectives:

- To define a one-one function, an onto function and the inverse of a function
- To study geometrical characterization of one-one and onto functions $f: \boldsymbol{R} \to \boldsymbol{R}$
  AND
- To practice the related problems

A function $f: A \to B$ is said to be **one-to-one** if distinct elements in the domain A have distinct images. In other words,

f is one-to-one if $f(a) = f(a') \Rightarrow a = a'$

Example: If $A = \{4,5,6\}$, $B = \{a, b, c, d\}$ and if $f: A \to B$ such that $f = \{(4, a), (5, b), (6, c)\}$ then f is one-to-one.

Example: The mapping $f: \boldsymbol{R} \to \boldsymbol{R}$ such that $f(x) = x^2$ is not a one-to-one function, since $f(-2) = 4$ and $f(2) = 4$, that is, two distinct elements −2 and 2 have the same image 4.

If a function $f: A \to B$ is such that two or more elements a_1, a_2 ….. of A have the same f- image in B, then the mapping is called **many-to-one mapping** or **many-to-one function**.

Example: If $A = \{a, b, c, d, e\}$, $B = \{1,2,3\}$, and if $f: A \to B$ is such that $f = \{(a, 1), (b, 1), (c, 1), (d, 2), (e, 2)\}$ then the function is many-to-one.

A function $f: A \to B$ is called an **onto** function if every element of B is the image of at least one element of A. For every $b \in B$ there exist at least one element $a \in A$ such that $f(a) = b$. That is,

f is onto if $\forall\, b \in B, \exists\, a \in A$ such that $f(a) = b$

In such a case, we say that f maps A onto B (Here the symbol $\forall$ means *for every*, and $\exists$ means *there exist*).

If $f: A \to B$ is not an onto function, that is, some of the elements of B remain unused, then f is called an **into** function.

Example: The function $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = |x|$ is not onto since the negative numbers in the co-domain are not used. Similarly, $f(x) = x^2$ is also not onto.

If $f: A \to B$ is both one-to-one and onto, then f is called a ***one-to-one correspondence*** between A and B. This terminology comes from the fact that each element of A will correspond to a unique element of B and vice versa.

We also use the term ***injective*** for a one-to-one function, ***surjective*** for an onto function, and ***bijective*** for a one-to-one correspondence.

Example: For the function $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = e^x$, which of the following is true.
(a) onto (b) many- to-one (c) one-to-one and into
(d) many-to- one and onto

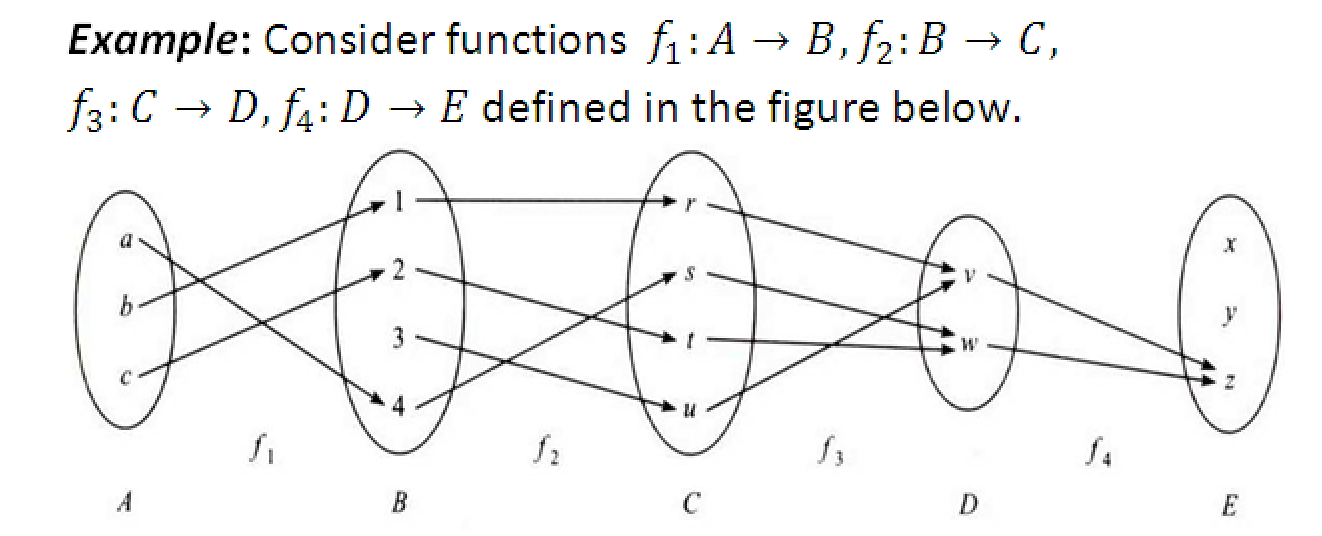
Solution: Let $x_1, x_2 \in \boldsymbol{R}$,
$f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$;
Therefore, f is one - one. Notice that e^x is always positive, i.e., the negative real numbers and zero have no preimages. Therefore, f is into.
Answer is (c).

A function $f: A \to B$ is said to be ***invertible*** if its inverse relation f^{-1} is a function from B to A. Equivalently, $f: A \to B$ is **invertible** if there exists a function $f^{-1}: B \to A$, called the **inverse** of f, such that

$$f^{-1} \circ f = I_A \text{ and } f \circ f^{-1} = I_B$$

In general, an inverse function f^{-1} need not exist or, equivalently, the inverse relation f^{-1} may not be a function.

Theorem: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.
Example: Consider functions $f_1: A \to B, f_2: B \to C$, $f_3: C \to D, f_4: D \to E$ defined in the figure below.



The function f_1 is one-to-one since no element of B is the image of more than one element of A. It is not onto since $3 \in B$ is not the image of any element of A under f_1.
The function, f_2 is one-to-one and f_2 is onto, since every element of C is the image of some element B under f_2. Further, f_3 is not one-to-one but onto and f_4 is neither one-to-one nor onto.

Since f_2 is both one-to-one and onto, it is a one-to-one correspondence between A and B. Hence f_2 is invertible and ${f_2}^{-1}$ is a function from C to B.

Geometrical Characterization

Consider a real-valued function $f: \boldsymbol{R} \to \boldsymbol{R}$. It may be identified with its graph which is plotted in the Cartesian plane $\boldsymbol{R}^2$. The concepts of being one-to-one and onto have the following geometrical meaning.

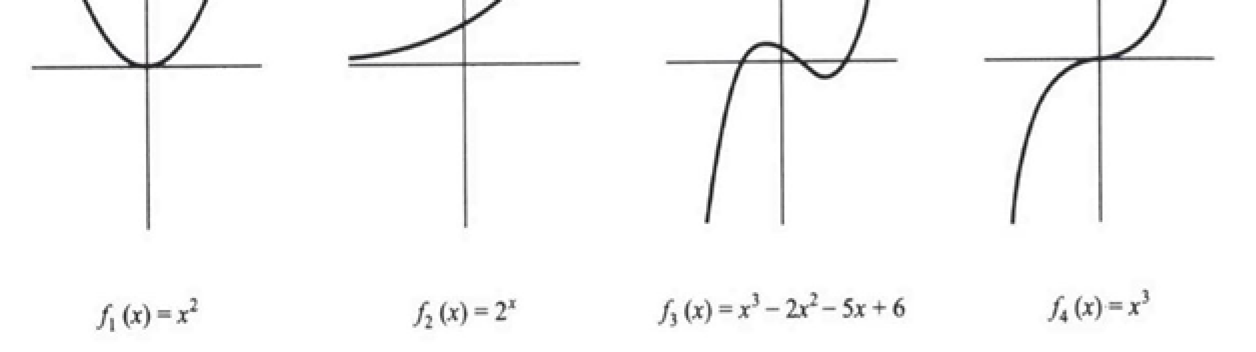
One-to-one means that there are no two distinct pairs (a_1, b) and (a_2, b) in f; hence each horizontal line in $\boldsymbol{R}^2$ can intersect the graph of f in *at most one point* (Horizontal line test).

Onto means that for every $b \in \boldsymbol{R}$ there is at least one point $a \in \boldsymbol{R}$ such that (a, b) belongs to the graph of f; hence each horizontal line in $\boldsymbol{R}^2$ must intersect the graph of f *at least once*.

Accordingly, the function $f: \boldsymbol{R} \to \boldsymbol{R}$ is one-to-one and onto (and therefore invertible) if and only if each horizontal line in $\boldsymbol{R}^2$ will intersect the graph of f in *exactly one point*.

Example: Consider the following four functions from $\boldsymbol{R}$ into $\boldsymbol{R}$ whose graphs are shown below.

$f_1(x) = x^2, f_2(x) = 2^x, f_3(x) = x^3 - 2x^2 - 5x + 6$,
$f_4(x) = x^3$



$f_1(x) = x^2$ $f_2(x) = 2^x$ $f_3(x) = x^3 - 2x^2 - 5x + 6$ $f_4(x) = x^3$

There are horizontal lines which intersect the graph of f_1 twice and there are horizontal lines which do not intersect the graph of f_1 at all; hence f_1 is neither one-to-one nor onto.
Similarly, f_2 is one-to-one but not onto, f_3 is onto but not one-to-one, and f_4 is both one-to-one and onto.
The inverse of f_4 is the cube root function, that is,

$${f_4}^{-1}(x) = \sqrt[3]{x}$$

NOTE:
Sometimes, we restrict the domain and co-domain of a function f in order to obtain an inverse function f^{-1}. For example, suppose we restrict the domain and co-domain of the function $f_1(x) = x^2$ to be the set D of nonnegative real numbers. Then f_1 is one-to-one and onto and its inverse is the square root function, that is,

$${f_1}^{-1}(x) = \sqrt{x}$$